1. Let $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -3 & 5 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{pmatrix}$.

   (a) What is the determinant of A?

   (b) What is the determinant of A^{-1}?

   (c) What is the determinant of A^{15}?

   (d) What is the determinant of A^T?

   (e) What is the value of the cofactor C_{34}?

   (f) How many nonzero terms are there in the permutation formula for the determinant of A?

   (g) Does the linear transformation A preserve or reverse orientation, or neither?

   (h) What is the characteristic polynomial of A?

   (i) Find a diagonal matrix Λ such that A is similar to Λ.

   (j) What are the eigenvalues of A^{-1}?

   (k) What are the eigenvalues of A^{15}?

   (l) What are the eigenvalues of A^T?

   (m) Is the matrix A^TA positive semi-definite, positive definite, or neither?

   Solution:

   (a) Since A is diagonal, the determinant of A is the product of the diagonal entries, so
   $$\det A = (2)(1)(-2)(-3) = 12.$$

   (b) $\det A^{-1} = 1/\det A = 1/12$.

   (c) $\det A^{15} = (\det A)^{15} = 12^{15}$.

   (d) $\det A^T = \det A = 12$

   (e) $C_{34} = (-1)^{3+4} \det \begin{pmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = -2$.

   (f) The only permutation matrix with all nonzero entries on or below the diagonal is the identity matrix, so the permutation formula for the determinant of A has only one nonzero entry.

   (g) Since $\det A > 0$, the linear transformation A preserves orientation.

   (h) The characteristic polynomial of A is $\det(A - \lambda I) = (2 - \lambda)(1 - \lambda)(-2 - \lambda)(-3 - \lambda)$.

   (i) A is similar to the doagonal matrix $\Lambda = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$.

   (j) The eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A: $1/2$, 1, $-1/2$, and $-1/3$.

(k) The eigenvalues of A^{15} are the 15th powers of the eigenvalues of A: 2^{15}, 1, $(-2)^{15}$, and $(-3)^{15}$.

(l) The eigenvalues of A^T are the same as the eigenvalues of A: 2, 1, -2, and -3.

(m) Since A has independent columns, the matrix A^TA is positive definite.

2. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal basis for $\mathbb{R}^3$, and suppose these are eigenvectors for a matrix A with eigenvalues $\lambda_1 = 7$, $\lambda_2 = 0$, and $\lambda_3 = -2$.

   (a) Describe a diagonalization for A, or explain why A is not diagonalizable.
   (b) Is A symmetric, positive semi-definite, positive definite, or none of these?
   (c) Is A invertible? Explain how you know.
   (d) Suppose $\mathbf{x} = 3\mathbf{v}_1 - 6\mathbf{v}_2$. Write $A\mathbf{x}$ as a linear combination of the eigenvectors of A.
   (e) Suppose $\mathbf{x} = 3\mathbf{v}_1 - 6\mathbf{v}_2$. Show that $\mathbf{x} \perp \mathbf{v}_3$.
   (f) Find a nonzero vector $\mathbf{x}$ such that $\mathbf{x}^T A\mathbf{x} = 0$ or explain why it is not possible.
   (g) Find a nonzero vector $\mathbf{x}$ such that $\mathbf{x}^T A\mathbf{x} < 0$ or explain why it is not possible.
   (h) What is the rank of A?
   (i) Find an orthonormal basis for the null space of A.
   (j) Find an orthonormal basis for the column space of A.

   Solution:

   (a) If $X = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ then $A = X\Lambda X^{-1}$. Since X is an orthogonal matrix, $A = X\Lambda X^T$.

   (b) Since Λ is diagonal, $\Lambda^T = \Lambda$, so

$$A^T = (X\Lambda X^T)^T = (X^T)^T \Lambda^T X^T = X\Lambda X^T = A.$$

   Thus A is symmetric.

   (c) Since $A\mathbf{v}_2 = 0\mathbf{v}_2 = \mathbf{0}$, $\mathbf{v}_2$ is a nontrivial element of $N(A)$, A is not invertible. [Or since $\det A$ is the product of the eigenvalues, $\det A = 0$. Or many other equivalent explanations.]

   (d) Using linearity,

$$A\mathbf{x} = A(3\mathbf{v}_1 - 6\mathbf{v}_2) = 3(A\mathbf{v}_1) - 6(A\mathbf{v}_2) = 3(7\mathbf{v}_1) - 6(0\mathbf{v}_2) = 21\mathbf{v}_1.$$

   (e) To show that $\mathbf{x} \perp \mathbf{v}_3$ we show their dot product is 0. Using linearity,

$$\mathbf{v_3}^T\mathbf{x} = \mathbf{v_3}^T(3\mathbf{v}_1 - 6\mathbf{v}_2) = 3\mathbf{v_3}^T\mathbf{v}_1 - 6\mathbf{v_3}^T\mathbf{v}_2 = 3(0) - 6(0) = 0$$

   where $\mathbf{v_3}^T\mathbf{v}_1 = 0$ and $\mathbf{v_3}^T\mathbf{v}_2 = 0$ since the eigenvectors are orthonormal.

   (f) If $\mathbf{x} = \mathbf{v}_2$ then

$$\mathbf{x}^T A\mathbf{x} = \mathbf{v}_2^T A\mathbf{v}_2 = \mathbf{v}_2^T(0\mathbf{v}_2) = \mathbf{v}_2^T\mathbf{0} = 0.$$

   (g) If $\mathbf{x} = \mathbf{v}_3$ then

$$\mathbf{x}^T A\mathbf{x} = \mathbf{v}_3^T A\mathbf{v}_3 = \mathbf{v}_3^T(-2\mathbf{v}_3) = -2\mathbf{v}_3^T\mathbf{v}_3 = -2.$$

   where $\mathbf{v_3}^T\mathbf{v}_3 = 1$ since the eigenvectors are orthonormal.

   (h) The rank of A is 2.

   (i) An orthonormal basis for the null space of A is $\{\mathbf{v}_2\}$.

   (j) An orthonormal basis for the column space of A is $\{\mathbf{v}_1, \mathbf{v}_3\}$.

3. Let $\mathbf{v} = (-1, 0, 1, -1, 1)$, $\mathbf{w} = (-2, 1, 0, -2, 0)$ and $V = \text{span}\{\mathbf{v}, \mathbf{w}\}$.

   (a) What is the dimension of V?

   (b) Find a matrix A such that $V = C(A^T)$. What is the rank of A? What fundamental subspace is $V^\perp$? What is the dimension of $V^\perp$?

   (c) Let $\mathbf{x} \in V$ and $\{\mathbf{q}_1, \mathbf{q}_2\}$ be any orthonormal basis for V. Prove that $\mathbf{x} = \text{proj}_{\mathbf{q}_1}\mathbf{x} + \text{proj}_{\mathbf{q}_2}\mathbf{x}$. [Hint: Use the fact that $\{\mathbf{q}_1, \mathbf{q}_2\}$ is a basis for V to write an equation, then take the dot product of both sides with an appropriately chosen vector and use linearity.]

   (d) Find the projection of the vector $\mathbf{w}$ onto $\mathbf{v}$.

   (e) Use the Gram-Schmidt algorithm to find an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2\}$ for V.

   (f) Find the projection of the vector $\mathbf{x} = (1, 1, 1, 1, 1)$ onto V.

   (g) Find the projection of the vector $\mathbf{x} = (1, 1, 1, 1, 1)$ onto $V^\perp$.

Solution:

(a) Since $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, the dimension of V is 2.

(b) If $A = \begin{pmatrix} \mathbf{v}^T \\ \mathbf{w}^T \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & -1 & 1 \\ -2 & 1 & 0 & -2 & 0 \end{pmatrix}$, then $V = C(A^T)$.
The rank of A is 2. Since the null space is the orthogonal complement of the row space, $V^\perp = N(A)$. The dimension of $V^\perp$ is $n - r = 5 - 2 = 3$.

(c) Assume $\{\mathbf{q}_1, \mathbf{q}_2\}$ is an orthonormal basis for V. First note that since $\mathbf{q}_1$ and $\mathbf{q}_2$ are unit vectors,

$$\text{proj}_{\mathbf{q}_1}\mathbf{x} = (\mathbf{q}_1^T\mathbf{x})\mathbf{q}_1 \quad \text{and} \quad \text{proj}_{\mathbf{q}_2}\mathbf{x} = (\mathbf{q}_2^T\mathbf{x})\mathbf{q}_2.$$

Since $\{\mathbf{q}_1, \mathbf{q}_2\}$ is a basis for V, $\mathbf{x}$ can be written as a linear combination of $\mathbf{q}_1$ and $\mathbf{q}_2$ in exactly one way:

$$\mathbf{x} = c_1\mathbf{q}_1 + c_2\mathbf{q}_2. \tag{1}$$

Thus it suffices to show that

$$c_1 = \text{proj}_{\mathbf{q}_1}\mathbf{x} = (\mathbf{q}_1^T\mathbf{x})\mathbf{q}_1$$

and

$$c_2 = \text{proj}_{\mathbf{q}_2}\mathbf{x} = (\mathbf{q}_2^T\mathbf{x})\mathbf{q}_2.$$

Taking the dot product of each side of Equation 1 with $\mathbf{q}_1$, we get

$$\mathbf{q}_1^T\mathbf{x} = \mathbf{q}_1^T(c_1\mathbf{q}_1 + c_2\mathbf{q}_2) = c_1\mathbf{q}_1^T\mathbf{q}_1 + c_2\mathbf{q}_1^T\mathbf{q}_2 = c_1(1) = c_2(0) = c_1.$$

Similarly, taking the dot product of each side of Equation 1 with $\mathbf{q}_2$, we get

$$\mathbf{q}_2^T\mathbf{x} = \mathbf{q}_2^T(c_1\mathbf{q}_1 + c_2\mathbf{q}_2) = c_1\mathbf{q}_2^T\mathbf{q}_1 + c_2\mathbf{q}_2^T\mathbf{q}_2 = c_1(0) = c_2(1) = c_2.$$

Thus $c_1 = \mathbf{q}_1^T\mathbf{x}$ and $c_2 = \mathbf{q}_2^T\mathbf{x}$.

(d)

$$\text{proj}_{\mathbf{v}}\mathbf{w} = \frac{\mathbf{v}^T\mathbf{w}}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = \frac{4}{4}\mathbf{v} = \mathbf{v} = (-1, 0, 1, -1, 1).$$

(e) First vector: Let $\mathbf{A}_1 = \mathbf{v}$. Then

$$\mathbf{q}_1 = \frac{\mathbf{A}_1}{\|\mathbf{A}_1\|} = \frac{1}{2}\mathbf{A}_1 = (1/2)(-1, 0, 1, -1, 1) = (-1/2, 0, 1/2, -1/2, 1/2).$$

Second vector:

$$\mathbf{A}_2 = \mathbf{w} - \text{proj}_{\mathbf{v}}\mathbf{w} = (-2, 1, 0, -2, 0) - (-1, 0, 1, -1, 1) = (-1, 1, -1, -1, -1).$$

Then

$$\mathbf{q}_2 = \frac{\mathbf{A}_2}{\|\mathbf{A}_2\|} = \frac{1}{\sqrt{5}}\mathbf{A}_2 = (1/\sqrt{5})(-1, 1, -1, -1, -1).$$

(f) The projection of the vector $\mathbf{x} = (1, 1, 1, 1, 1)$ onto V is

$$\begin{aligned}
\text{proj}_{\mathbf{q}_1}\mathbf{x} + \text{proj}_{\mathbf{q}_2}\mathbf{x} &= (\mathbf{q}_1^T\mathbf{x})\mathbf{q}_1 + (\mathbf{q}_2^T\mathbf{x})\mathbf{q}_2 \\
&= (0)\mathbf{q}_1 + (-3/\sqrt{5})\mathbf{q}_2 \\
&= (-3/\sqrt{5})(1/\sqrt{5})(-1, 1, -1, -1, -1) \\
&= (-3/5)(-1, 1, -1, -1, -1)
\end{aligned}$$

(g) Since $\mathbf{x} = \mathbf{p} + \mathbf{e}$, where $\mathbf{p}$ is the projection of $\mathbf{x}$ onto V and $\mathbf{e}$ is orthogonal to $\mathbf{p}$, the projection of the vector $\mathbf{x} = (1, 1, 1, 1, 1)$ onto $V^\perp$ is

$$\mathbf{x} - \mathbf{p} = (1, 1, 1, 1, 1) - (-3/5)(-1, 1, -1, -1, -1) = (2/5, 8/5, 2/5, 2/5, 2/5).$$